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On the induced transverse gauge invariant mass term

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Abstract

We derive a general expression for the gauge invariant mass (m_G) for an Abelian gauge field, as induced by vacuum polarization, in 1 + 1 dimensions. From its relation to the chiral anomaly, we show that m_G has to satisfy a certain dynamical quantization condition. This quantization can, on the other hand, be explicitly verified by using the exact general expression for the gauge invariant mass in terms of the fermion propagator. This result is applied to some explicit examples, exploring the possibility of having interesting physical situations where the value of m_G departs from its canonical value of $\frac{e^2}{\pi}$ for a single massless Dirac fermion in 1 + 1 dimensions. We also study the possibility of generalizing the results to the (2 + 1)-dimensional case at finite temperature, showing that there are indeed situations where a finite and non-vanishing gauge invariant mass is induced.

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1. Introduction

Some important physical quantities displaying quantization properties, may sometimes be represented by means of *momentum* space integrals which exhibit their topologically invariant character. Considerable effort has been devoted to find these representations, since they are often very useful to prove their quantization, as well as their stability under perturbations.

This is the case, for instance, of the transverse conductivity σ in QED₃, which can be represented as a momentum space integral given by [1, 2]

$$\sigma = \frac{1}{3!} i e^2 \varepsilon^{\mu\nu\rho} \int \frac{d^3 p}{(2\pi)^3} \text{Tr}[\partial_\mu S^{-1} S \partial_\nu S^{-1} S \partial_\rho S^{-1} S] \quad (1)$$

where $S(p)$ is the full fermion propagator. As shown in [1], due to the ultraviolet behaviour of the fermion propagator, this integral reduces to the Kronecker topological invariant which

labels the homotopy classes in $\Pi_2(S^2)$, that is, σ is quantized. A similar representation for the induced Chern–Simons coefficient has been obtained in [3] by using a cubic lattice regularization for the Euclidean fermionic action. In this case, the possible values for the Chern–Simons coefficient turn out to be labelled by the winding number characterizing the mapping between the three-dimensional torus in momentum space and the normalized quaternion corresponding to the fermion propagator: $S(p)/\sqrt{\det(S)}$.

Topology in momentum space has also been advocated in [4], in order to discuss the stability of neutrino masses in the standard model and the spectrum of excitations in effective two-dimensional models such as helium-3. On the other hand, in the context of Yang–Mills theories, one of the main open problems is that of understanding the non-perturbative generation of a mass gap for the gauge fields as a consequence of the dynamics in the infrared regime. This problem is of crucial importance for confinement in QCD, and for analysing finite temperature effects. For instance, in [5], a gap equation, based on the introduction of gauge invariant mass terms, has been proposed and applied to the Yang–Mills theory in $2 + 1$ dimensions.

The aim of this work is to obtain a useful momentum space representation for detecting the existence of an induced gauge invariant mass (m_G) for an Abelian gauge field. In particular, we shall be able to derive a general expression for m_G , valid for different spacetime dimensions and geometries in momentum space.

This paper is organized as follows. Section 2 is devoted to a short review of the well-known relationship between the induced gauge invariant mass and the chiral anomaly in $1 + 1$ dimensions. In section 3, a general momentum space representation for the induced gauge invariant mass is provided. Section 4 is devoted to the applications of the aforementioned representation to the case of square lattice geometry and to the case of finite temperature in $2 + 1$ dimensions. Section 5 presents our conclusions.

2. Relationship between the gauge invariant mass and the chiral anomaly in $1 + 1$ dimensions

By gauge invariant mass, m_G , we understand the value of the constant appearing in a gauge invariant mass term for an Abelian gauge field. In $(d + 1)$ -dimensional Euclidean spacetime, a gauge invariant mass term Lagrangian \mathcal{L}_m is given explicitly by

$$\mathcal{L}_m = \frac{1}{2} m_G^2 A_\mu \delta_{\mu\nu}^\perp A_\nu \quad (2)$$

where $\delta_{\mu\nu}^\perp = \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}$ is the transverse Kronecker δ in $D = d + 1$ dimensions.

This mass term for A_μ is explicitly gauge invariant, although, of course, at the price of introducing a non-locality in the action. That is often the reason for not including this term in the classical Lagrangian of a standard local quantum field theory. In spite of this fact, terms like this naturally arise when evaluating radiative corrections to the effective action. In *massless* QED in $1 + 1$ dimensions, i.e. the Schwinger model [6], it is induced by the one-loop vacuum polarization graph. Moreover, for such a model, the one-loop result is exact, since it does not suffer from higher order corrections. These results can also be shown to be related to the chiral anomaly in $1 + 1$ dimensions, whence the non-renormalization of m_G is inherited. For this model, the non-locality can be made good since, in the bosonization approach, the gauge field may be written in terms of two scalar fields: $A_\mu = \epsilon_{\mu\nu} \partial_\nu \sigma + \partial_\mu \varphi$. In terms of these scalars, the gauge invariant mass term becomes a standard, local, mass term for the σ field.

In the $(1 + 1)$ -dimensional case, the value of m_G for a single fermionic flavour has the well-known value $m_G^2 = e^2/\pi$. In this paper, we are interested in deriving general expressions for m_G , as a first step towards extending the useful $(1 + 1)$ -dimensional result into several

non-trivial directions. In order to define the problem more precisely, we write m_G in terms of the exact vacuum polarization tensor³ $\tilde{\Pi}_{\mu\nu}(k)$. The Ward identity for $\tilde{\Pi}_{\mu\nu}(k)$, $k_\mu \tilde{\Pi}_{\mu\nu}(k) = 0$, allows us to write

$$\tilde{\Pi}_{\mu\nu}(k) = \tilde{\Pi}(k^2)\delta_{\mu\nu}^\perp(k) \tag{3}$$

where $\tilde{\Pi}(k^2)$ is a scalar function, and $\delta_{\mu\nu}^\perp(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$. Thus, m_G may also be obtained as

$$m_G^2 = \lim_{k \rightarrow 0} \tilde{\Pi}(k^2). \tag{4}$$

This definition implicitly assumes gauge and Lorentz invariance, two conditions that, except for explicit indication to the contrary, shall be maintained in everything that follows.

We shall be first concerned with models that can be described by an Euclidean action with the following structure:

$$S = \int d^D x \mathcal{L} \quad \mathcal{L} = \mathcal{L}_F + \mathcal{L}_G \tag{5}$$

where \mathcal{L}_F and \mathcal{L}_G denote the fermion and gauge field Lagrangians, respectively.

Let us begin with a consideration of the (1 + 1)-dimensional case, in the simplest non-trivial situation of massless QED(1 + 1). This will amount to a re-derivation of known results, although we shall present them here from a different perspective. The gauge field Lagrangian is

$$\mathcal{L}_G = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

while fermionic matter is described by a massless Dirac field ψ , with a Lagrangian

$$\mathcal{L}_F = \bar{\psi}(\not{\partial} + ie \not{A})\psi \tag{7}$$

where we adopted the conventions

$$\begin{aligned} (\gamma_\mu)^\dagger &= \gamma_\mu & \gamma_5^\dagger &= \gamma_5 & \gamma_\mu \gamma_\nu &= g_{\mu\nu} + i\epsilon_{\mu\nu} \gamma_5 \\ g_{\mu\nu} &= \delta_{\mu\nu} & \epsilon_{01} &= +1. \end{aligned} \tag{8}$$

The exact vacuum polarization tensor $\tilde{\Pi}_{\mu\nu}$ is, as usual, defined in terms of the connected gauge field two-point function $G_{\mu\nu}$ by

$$\langle A_\mu A_\nu \rangle_{\text{conn}} = G_{\mu\nu} \quad G_{\mu\nu}^{-1}(k) = k^2 \delta_{\mu\nu}^\perp + \tilde{\Pi}_{\mu\nu}(k). \tag{9}$$

It is easy to see that there is, indeed, a relation between m_G and the chiral anomaly $\mathcal{G}(A)$, since the former may actually be derived from the latter, at least for massless QED in 1 + 1 dimensions. To that end, we define $\mathcal{G}(A)$ by

$$\partial_\mu \langle J_\mu^5 \rangle = \mathcal{G}(A) \tag{10}$$

where $\langle J_\mu^5 \rangle$ denotes the quantum average of the axial current $J_\mu^5 \equiv -ie \bar{\psi} \gamma_5 \gamma_\mu \psi$ in the presence of an external gauge field \mathcal{A}_μ (we use the script \mathcal{A}_μ notation to distinguish it from the *dynamical* gauge field A_μ). Hence, the integrated form of the anomaly [7] is

$$\int d^2 x \mathcal{G}(A) = \int d^2 x \partial_\mu \langle J_\mu^5 \rangle = \int d^2 x \epsilon_{\mu\nu} \partial_\mu \langle J_\nu \rangle \tag{11}$$

where we used the relation $J_\mu^5 = \epsilon_{\mu\nu} J_\nu$, and $J_\mu \equiv e \bar{\psi} \gamma_\mu \psi$ is the vector current. Then, from our knowledge of the chiral anomaly, we may use the fact that $\mathcal{G}(A)$ is linear in \mathcal{A} to adopt the linear approximation for $\langle J_\nu \rangle$ in (11), namely

$$\langle J_\mu(x) \rangle = \int d^2 y \Pi_{\mu\nu}(x - y) \mathcal{A}_\nu(y) \tag{12}$$

³ The tilde denotes the momentum space version of an object whenever it is convenient to distinguish it from its coordinate space version.

with $\Pi_{\mu\nu}(x-y)$ denoting the coordinate space vacuum polarization function. It should be noted that the $\Pi_{\mu\nu}(x-y)$ appearing in (12) is exact, since only an expansion in the external field has been performed.

Then, inserting (12) into (11) and Fourier transforming, we find that

$$\int d^2x \mathcal{G}(\mathcal{A}) = \lim_{k \rightarrow 0} \tilde{\Pi}(k^2) \lim_{k \rightarrow 0} \epsilon_{\mu\nu} i k_\mu \tilde{\mathcal{A}}_\nu(k) \quad (13)$$

assuming that both limits exist. Taking (4) into account, this amounts to

$$\int d^2x \mathcal{G}(\mathcal{A}) = m_G^2 \Phi(\mathcal{A}) \quad (14)$$

where $\Phi(\mathcal{A}) = \int d^2x \epsilon_{\mu\nu} \partial_\mu \mathcal{A}_\nu$ is the total flux of $\epsilon_{\mu\nu} \partial_\mu \mathcal{A}_\nu$ in Euclidean spacetime. Equation (14) yields m_G in terms of the anomaly. Of course, the left-hand side in equation (14), being the spacetime integral of the anomaly, will have the form of a coefficient times $\Phi(\mathcal{A})$, namely

$$\int d^2x \mathcal{G}(\mathcal{A}) = \xi \Phi(\mathcal{A}) \quad (15)$$

thus the content of (14) is that the coefficient $\xi = e^2/2\pi$ gives precisely the value of m_G^2 . Now, only configurations with $e^2\Phi(\mathcal{A}) \in \mathbb{Z}$ may have a finite action, and this implies that ξ could have only been $e^2/2\pi$ or an integer multiple of this quantity, i.e.,

$$m_G^2 = k \frac{e^2}{2\pi} \quad k = 0, 1, 2, \dots \quad (16)$$

Equations (14) and (16) had not been, to the authors' knowledge, previously derived in the literature.

It must of course be possible to prove the quantization condition (16), starting from the fermionic side of the problem. Indeed, this is the case for the one-loop approximation, since we also know that, to that order, the integral of the anomaly is in fact the index of the Dirac operator, namely

$$\int d^2x \mathcal{G}(\mathcal{A}) = \frac{1}{2\pi} (n_+ - n_-) \quad (17)$$

where n_\pm denotes the number of zero modes of positive and negative chirality in the given background. In our conventions, it is easy to realize that $n_+ - n_- = e^2\Phi(\mathcal{A})$.

In the case in which a current-current interaction is introduced, the corresponding effect can be evaluated by means of the bosonization rules, which give

$$\partial_\mu \langle J_\mu^5 \rangle = \frac{e^2}{\pi + g} \epsilon_{\mu\nu} \partial_\mu A_\nu \quad (18)$$

where g denotes the current-current coupling constant. This relationship would imply, through equation (14), a renormalized gauge invariant mass, in agreement with the result of [9].

Also, when the fermions are massive, the right-hand side of equation (10) has to be supplemented with the additional term $-2m\bar{\psi}\gamma_5\psi$ coming from the explicit chiral symmetry breaking. Accordingly, equation (13) reads

$$\int d^2x \partial_\mu \langle J_\mu^5 \rangle = \int d^2x (\mathcal{G}(\mathcal{A}) - 2m\langle \bar{\psi}\gamma_5\psi \rangle) = \lim_{k \rightarrow 0} \tilde{\Pi}(k^2) \lim_{k \rightarrow 0} \epsilon_{\mu\nu} i k_\mu \tilde{\mathcal{A}}_\nu(k). \quad (19)$$

Due to the short-range behaviour of the massive Dirac fields, the left-hand side identically vanishes, which together with equation (4) gives a zero gauge invariant mass m_G . In particular, this implies that the contribution of the chiral anomaly cancels against that of the explicit chiral symmetry breaking term [10].

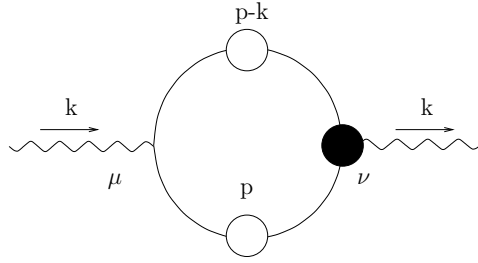


Figure 1. The exact vacuum polarization graph.

3. Momentum space representation for the gauge invariant mass

We try, in what follows, to use the fermionic point of view exclusively, in an attempt to derive the previous results therefrom, without assuming that a loop expansion has been performed. To that end, let us derive an exact expression for m_G in terms of the fermion propagator. We start from the (D -dimensional) general expression for $\tilde{\Pi}_{\mu\nu}$, as given by the Schwinger–Dyson equations [8], which amounts to the diagram of figure 1, where the white circles show that the lines are full fermion propagators, while the black circle represents the full vertex functions. The external legs are, of course, to be truncated, but we have drawn them for the sake of clarity. The analytic expression corresponding to figure 1 is

$$\tilde{\Pi}_{\mu\nu}(k) = -e^2 \int \frac{d^D p}{(2\pi)^D} \text{Tr}[\gamma_\mu S_F(p) \Gamma_\nu(p, p - k) S_F(p - k)] \quad (20)$$

where S_F is the momentum propagator and Γ_ν is the exact vertex function. Noting that $\Pi_{\mu\nu}$ may be expressed as in (3), we see that

$$m_G^2 = \tilde{\Pi}(0) = (D - 1)^{-1} \lim_{k \rightarrow 0} \tilde{\Pi}_{\mu\mu}(k) \quad (21)$$

or

$$m_G^2 = -e^2 (D - 1)^{-1} \lim_{k \rightarrow 0} \int \frac{d^D p}{(2\pi)^D} \text{Tr}[\gamma_\mu S_F(p) \Gamma_\mu(p, p - k) S_F(p - k)]. \quad (22)$$

Since the theory is gauge invariant, we may relate $\lim_{k \rightarrow 0} \Gamma_\mu(p, p - k)$ to the fermion propagator, by using the exact Ward identity

$$\lim_{k \rightarrow 0} \Gamma_\mu(p, p - k) = -i \frac{\partial}{\partial p_\mu} S_F^{-1}(p). \quad (23)$$

This identity may be used in (22) to obtain

$$\begin{aligned} m_G^2 &= ie^2 (D - 1)^{-1} \int \frac{d^D p}{(2\pi)^D} \text{Tr} \left[\gamma_\mu S_F(p) \frac{\partial}{\partial p_\mu} S_F^{-1}(p) S_F(p) \right] \\ &= -ie^2 (D - 1)^{-1} \int \frac{d^D p}{(2\pi)^D} \frac{\partial}{\partial p_\mu} \text{Tr}[\gamma_\mu S_F(p)] \end{aligned} \quad (24)$$

which is the most general expression we shall continue to work with. It should be noted that (24) is still formal, in the sense that we have not yet made explicit any regularization method.

Before exploring its divergences in the general case, we note that, to one-loop order, they are the well-known divergences in the vacuum polarization graph of figure 2.

In $(1 + 1)$, the expression for m_G suffers from both UV and IR divergences, as is clear from (24), since, when applied to the free Dirac propagator $S_F^{(0)} = -i \not{p}^{-1}$, it yields

$$m_G^2 = -2e^2 \int \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_\mu} \left(\frac{p_\mu}{p^2} \right). \quad (25)$$

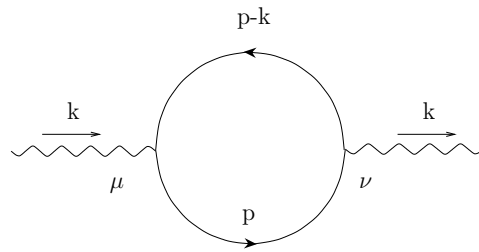


Figure 2. The one-loop vacuum polarization graph.

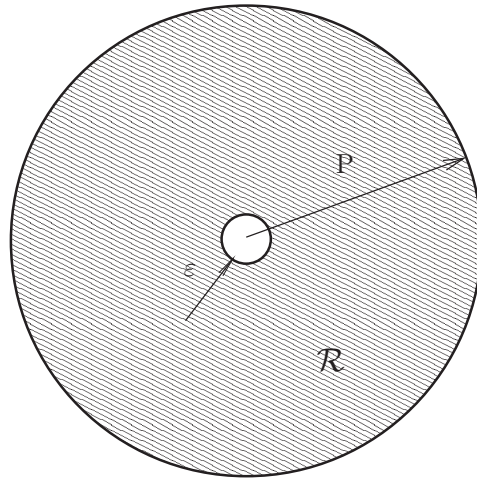


Figure 3. The integration region for the integrals in (26).

In order to make sense of this expression, we exclude a circle of radius ε around the origin of the momentum plane to avoid the IR singularity, and also use a Pauli–Villars regulator to tame the UV divergences. This amounts to defining

$$m_G^2 = -2e^2 \int_{\mathcal{R}} \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_\mu} \left(\frac{p_\mu}{p^2} \right) + 2e^2 \int_{\mathcal{R}} \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_\mu} \left(\frac{p_\mu}{p^2 + \Lambda^2} \right) \quad (26)$$

where \mathcal{R} is the region illustrated in figure 3 and Λ denotes the mass of the regulator field. The momentum P denotes the radius of the integration region and, of course, $P \rightarrow \infty$ since the theory is already regularized.

By a straightforward application of the two-dimensional Gauss' theorem, we may convert these integrals of divergences into fluxes of radial vector fields. This procedure yields

$$m_G^2 = -\frac{2e^2}{(2\pi)^2} \left[-2\pi \frac{\varepsilon}{\varepsilon} + 2\pi \frac{\varepsilon}{\varepsilon^2 + \Lambda^2} \right] \quad (27)$$

which in the $\varepsilon \rightarrow 0$ limit becomes

$$m_G^2 = \frac{e^2}{\pi} \quad (28)$$

as it should. The procedure we have followed has a simple electrostatic analogy: m_G is given by the integral of the divergence of an 'electric field', so by following this analogy it is

proportional to the total ‘electric charge’. Because of the regulator, though, only the charges at the origin are relevant; namely, the subtraction due to the UV regulator leads to

$$m_G^2 = \frac{ie^2}{(2\pi)^2} \oint_{\mathcal{C}(\varepsilon)} dl \hat{n}_\mu \text{Tr}[\gamma_\mu S_F(p)] \quad (29)$$

where the integral is taken along a small curve of radius ε enclosing the origin. \hat{n}_μ denotes the outer normal to $\mathcal{C}(\varepsilon)$. In what follows, we shall argue that this kind of expression can be generalized to the full theory. Indeed, we can always say that the role of *any* UV regularization will be to modify the large momentum behaviour of the propagator in such a way that all the points with infinite momentum can be identified. For the Pauli–Villars case, this can be shown to hold simply by combining the contributions of both fermion propagators into one integral, to define a regularized propagator. Namely

$$m_G^2 = -2e^2 \int_{\mathcal{R}} \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_\mu} \left[\frac{\Lambda^2}{p^2(p^2 + \Lambda^2)} p_\mu \right] \quad (30)$$

which contains the divergence (in momentum space) of a vector field which decreases as $\sim p^{-4}$ at infinity.

This will be one of the properties we shall demand of a regularization at any finite order of the loop expansion, namely, the points at infinity in the momentum space may, from the point of view of this calculation, be identified. The other condition is that, in the small momentum region, i.e. momenta much smaller than the cut-off, the behaviour of the propagator should be the same as for the unregularized propagator. Thus, only the small region around zero may contribute. This, in turn, will produce a non-vanishing answer only when the field is massless, as we shall see now.

Assuming a standard Dirac action for the fermions, the general form of the fermion propagator in momentum space is, of course,

$$S_F^{-1}(p) = iA(p) \not{p} + B(p) \quad (31)$$

where A and B are real functions that depend only on the scalar p^2 . Then an application of (24) yields

$$\begin{aligned} m_G^2 &= -ie^2 \int \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_\mu} \text{Tr} \left[\gamma_\mu \frac{A(p) \not{p}}{A^2(p)p^2 + B^2(p)} \right] \\ &= \frac{e^2}{(2\pi)^2} \text{Tr}(I) \oint_{\mathcal{C}(\varepsilon)} dl \hat{n}_\mu \left[\frac{A(p)p_\mu}{A^2(p)p^2 + B^2(p)} \right] \end{aligned} \quad (32)$$

where $\text{Tr}(I)$ counts the dimension of the Dirac algebra representation. Evaluating the integral along $\mathcal{C}(\varepsilon)$

$$m_G^2 = \frac{e^2}{2\pi} \text{Tr}(I) \frac{A(\varepsilon)\varepsilon^2}{A^2(\varepsilon)\varepsilon^2 + B^2(\varepsilon)}. \quad (33)$$

Since $\varepsilon \rightarrow 0$, we may conclude from here that, if the fermion is massive, $B(0) \neq 0$, then $m_G = 0$. This is indeed the case for the massive Schwinger model. Regarding the massless case, we obtain

$$m_G^2 = \frac{e^2}{2\pi} \text{Tr}(I) \frac{1}{A(0)} = \frac{e^2}{2\pi} \text{Tr}(I) \quad (34)$$

where we have used the normalization condition $A(0) = 1$, which holds at any finite order of the loop expansion. This condition fixes the residue of the perturbative electron propagator

at the pole $p^2 = 0$. Strictly speaking, this normalization will change in an interacting theory. Indeed, the fermion propagator can, in general, be rewritten using the spectral representation [8]

$$S_F(p) = \int_0^\infty d\mu^2 \frac{-i\rho_1(\mu^2) \not{p} + \rho_2(\mu^2)}{p^2 + \mu^2} \quad (35)$$

where ρ_1 and ρ_2 are real functions. It is then clear, by the linearity of this expression, that there will be a finite gauge invariant mass if there is an isolated pole at zero momentum, namely, if we can write

$$S_F(p) = Z_2 \frac{-i\not{p}}{p^2} + \int_{m^2}^\infty d\mu^2 \frac{-i\rho_1(\mu^2) \not{p} + \rho_2(\mu^2)}{p^2 + \mu^2} \quad (36)$$

where m^2 is the multiparticle threshold. In this situation, we would obtain

$$m_G^2 = \frac{e^2}{2\pi} \text{Tr}(I) \frac{1}{A(0)} = \frac{e^2}{2\pi} \text{Tr}(I) Z_2 \quad (37)$$

where Z_2 is a constant, smaller than 1 because of the spectral condition

$$1 = Z_2 + \int_{m^2}^\infty d\mu^2 \rho_1(\mu^2). \quad (38)$$

Equation (37) may seem to contradict our remarks on the relation between m_G^2 and the anomaly, for example, (16). The resolution of this apparent paradox is that the anomaly, as we understood it in (10), is defined in terms of the divergence of the current. The matrix elements of this current, when evaluated through the reduction formulae, will require the introduction of a Z_2 factor ($Z_2^{1/2}$ for each fermionic field), thus the proper relation that generalizes (16) is

$$m_G^2 = k \frac{e^2}{2\pi} Z_2 \quad k = 0, 1, 2, \dots \quad (39)$$

We wish to point out, however, that this equation may be of difficult practical application, since it requires knowledge of the exact Z_2 . This exact constant is bounded by 1 but, as is well known, it may be divergent and gauge dependent if evaluated at a given order in perturbation theory. The possible gauge dependence of the exact Z_2 is a subject that certainly deserves further study and shall be discussed elsewhere. Of course, the ratio between m_G^2 and the exact Z_2 must indeed be gauge independent and quantized.

The original expression for m_G required a regularization procedure in order to be well defined. However, being determined by the low momentum behaviour of the propagator, m_G should be independent of the regularization. For example, for a generalized Pauli–Villars regularization in the one-loop case, one might include a set of N regulator fields and define

$$S_F^{\text{reg}}(p) = \sum_{n=0}^N C_n \frac{1}{i\not{p} + M_n} \quad (40)$$

where M_n denote the regularizing masses, and the index $n = 0$ is reserved for the unregularized propagator, which has $M_0 = 0$ and $C_0 = 1$. The masses M_n and the coefficients C_n for $n > 0$ are chosen in order to verify the desired behaviour in the UV. However, since only the behaviour around zero momentum is relevant, and all the regulators are massive, then just the unregularized massless propagator contributes.

4. Applications to different geometries and spacetime dimensions

4.1. The case of the $(1 + 1)$ -dimensional square lattice

It should be clear from what we have said above that the geometry of the momentum space is crucial in the problem of evaluating m_G . An interesting example of this is the case of a

lattice regularized theory. Indeed, assuming that the coordinate space has been discretized to an (infinite) square lattice with lattice spacing a , the momentum space becomes a (continuous) torus. The lattice points are then defined as the set of points $\sigma_\mu = at_\mu$ with $t_\mu \in \mathbb{Z}$ for $\mu = 0, 1$.

The naive (i.e. no Wilson term) lattice fermion propagator for a massive Dirac fermion then becomes

$$S_F(p) = \frac{a}{i\gamma_\mu C_\mu(ap) + am} \tag{41}$$

where $C_\mu(ap) = \sin(ap_\mu)$, and the p_μ are continuous variables (because the lattice is infinite), in the first Brillouin zone, namely

$$-\pi < p_\mu \leq \pi \quad \mu = 0, 1. \tag{42}$$

Then, the expression for m_G is

$$m_G^2 = -ie^2 \int_{\mathcal{B}} \frac{d^2p}{(2\pi)^2} \frac{\partial}{\partial p_\mu} \text{Tr} \left[\gamma_\mu \frac{a}{i\mathcal{Q}(ap) + am} \right] \tag{43}$$

where \mathcal{B} denotes the first Brillouin zone. The fact that we are dealing with a momentum torus is obvious because of the periodicity of the functions C_μ , a property which is preserved of course for more general propagators. But the functions C_μ will simultaneously vanish for the (p_0, p_1) values in the set

$$\{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\} \tag{44}$$

which contains not just the origin, but also three other unwanted points. The behaviour of the lattice propagator is similar for each of these points, in the sense that it has a single pole and in consequence there is a propagating Dirac fermion; however, the residue matrix at each pole is different and corresponds to a change in the sign of the corresponding γ matrices. Indeed, the residues are such that if we use γ_0 and γ_1 for the matrices at $(0, 0)$, then we have γ_0 and $-\gamma_1$ at $(0, \pi)$, $-\gamma_0$ and γ_1 at $(\pi, 0)$ and $(-\gamma_0, -\gamma_1)$ at (π, π) . Now, any one of these four sets of γ matrices yields the *same* answer for the gauge invariant mass, since they amount to equivalent representations of the Dirac algebra. In this case, since the propagator contains both chiralities, the unwanted particles ('doublers') are also vector like.

Then, when $m = 0$, an application of the Gauss law on the torus, with the circles around the four poles of the propagator excluded, yields four times the contribution of a single Dirac fermion, since the gauge invariant mass has the same sign for each doubler. This is just a manifestation of the 'doubling' problem of lattice fermions, related to the Nielsen–Ninomiya theorem [11], in this case applied for a magnitude which depends only on the real part of the effective action.

Thus

$$[m_G^2]_{\text{lattice}} = 4 \frac{e^2}{\pi}. \tag{45}$$

4.2. QED₃ at finite temperature

Another circumstance where the structure of momentum space allows for the emergence of a non-trivial gauge invariant mass, is the case of QED in 2 + 1 dimensions at finite temperature in the presence of 'large' A_0 field configurations [12]. Parity conserving versions of QED(2 + 1), like the model introduced by Dorey and Mavromatos [13], have been extensively studied as quantum field theory models at finite temperature [12]. Most of what we shall say about this here, however, holds true for both the parity conserving and the parity breaking cases.

We now discuss QED(2 + 1) with regard to only one of its aspects, namely, its gauge invariant mass. We recall that in finite temperature quantum field theory, Lorentz invariance is lost. Indeed, in the Matsubara formalism, which we shall adopt, the time coordinate runs from 0 to $\beta = \frac{1}{T}$. This lack of Lorentz invariance implies that one shall, in principle, have different masses for the spatial and temporal components of the gauge field. Namely, the natural extension of (2) to this case would be

$$\mathcal{L}_m = \frac{1}{2}m_{\text{el}}^2 A_0(x, \tau)A_0(x, \tau) + \frac{1}{2}m_{\text{mag}}^2 A_j(x, \tau)\delta_{jk}^\perp A_k(x, \tau) \quad (46)$$

where $\delta_{jk}^\perp = \delta_{jk} - \frac{\partial_j \partial_k}{\partial^2}$ is the transverse Kronecker δ for the two spatial dimensions, and τ is the imaginary time. The two components m_{el} and m_{mag} are the electric and magnetic masses, which can, of course, be different for $T \neq 0$. These two masses play the role of components of a ‘mass tensor’ for A_μ . We shall deal exclusively with the magnetic mass. It is not difficult to apply a similar derivation to the one used for the $T = 0$ case, to obtain

$$m_{\text{mag}}^2 = -ie^2 \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_j} \text{Tr} [\gamma_j S_F(p, n)] \quad (47)$$

where the sum runs over the Matsubara frequencies $\omega_n = (2n + 1)\pi T$ and S_F is the finite temperature fermion propagator.

For the free, massless fermion propagator in the presence of a large A_0 , we have [12]

$$S_F(p, n) = \frac{1}{i\gamma_0 \tilde{\omega}_n + i\gamma_k p_k} \quad (48)$$

where $\tilde{\omega}_n = \omega_n + eA_0$. A_0 is assumed to be constant, a fact which can always be achieved by a small gauge transformation. The value of this constant is of course determined by the quantity $\int d\tau A_0(\tau)$, where $A_0(\tau)$ is an arbitrary time-dependent configuration. Then

$$m_{\text{mag}}^2 = -ie^2 \frac{1}{\beta} \text{Tr}(I) \sum_{n=-\infty}^{+\infty} \int \frac{d^2 p}{(2\pi)^2} \frac{\partial}{\partial p_j} \frac{p_j}{\mathbf{p}^2 + \tilde{\omega}_n^2} \quad (49)$$

which looks like a series of (1 + 1)-dimensional contributions. Of course, whenever $\omega_n = -eA_0$, we have a non-vanishing contribution, of the same kind as those appearing in 1 + 1 dimensions. That condition on A_0 amounts to

$$e \int_0^\beta d\tau A_0(\tau) = (2n + 1)\pi. \quad (50)$$

Thus, we may write the result for m_{el}^2 as

$$m_{\text{mag}}^2 = \frac{e^2}{2\pi\beta} \text{Tr}(I) \sum_{n=-\infty}^{+\infty} \delta \left[e \int_0^\beta d\tau A_0(\tau), (2n + 1)\pi \right] \quad (51)$$

where we have used the notation $\delta[a, b]$ to denote the function which is equal to 1 when $a = b$, and $\delta[a, b] = 0$ when $a \neq b$ (a and b are not integers, in general).

The β dependence should have been expected by dimensional analysis, since in 2 + 1 dimensions, e^2 has the units of a mass. Regarding the existence of particular points where the magnetic mass is generated, the physical reason for that is that when the condition for A_0 is met, dimensional reduction occurs because there is a massless mode, and the reduced model is then tantamount to a Schwinger model.

5. Conclusions

In this work, a useful momentum space representation for the gauge invariant mass has been obtained and applied to different situations, namely, massless and massive QED in 1 + 1 dimensions, the (1 + 1)-dimensional square lattice and QED₃ at finite temperature. The key ingredient for this representation is the validity of gauge invariance expressed by the Ward identity, and of course its consistency with the dynamics defined by the Schwinger–Dyson equations.

It should be clear from (24) that the value of m_G^2 depends entirely on the infrared behaviour of the fermion propagator, resulting indeed from the infrared dynamics. In particular, the momentum representation for the gauge invariant mass in 1 + 1 dimensions turns out to be related to the chiral anomaly. This relationship suggests that m_G should possess some kind of stability against perturbations. For instance, the introduction of a current–current interaction amounts to a smooth change [9] of the coefficient $A(0)$ in equation (34), while preserving the vanishing of $B(p) = 0$. This means that, apart from a normalization factor, m_G^2 is still finite and non-vanishing.

A further comment we would like to add, and which is related to the previous discussion is one concerning the meaning of formula (24). The main difference between our momentum space representation for m_G and similar representations [1, 2, 4] lies in the presence of the factor $A(0)$. This is related to the fact that in our case the formula cannot be written just in terms of the $SU(2)$ projection of the fermion propagator, as in the other cases. Therefore, when including interactions, m_G is not protected from being renormalized. In this sense, the quantized gauge invariant mass of the (1 + 1)-dimensional Schwinger model is not stable against interactions. In the present case, what is stable against perturbations is the ratio between m_G and Z_2 . If perturbations are included in the massive Schwinger model or in the massless case, m_G will however still be $m_G = 0$ or $m_G \neq 0$, respectively.

We finish with a comment and outlook on the possible implications of our results to higher dimensional systems at zero temperature, where dimensional reduction does not occur. It should be noted that the main difficulty is the fact that m_G^2 , as given by (24), seems not to be interesting for the higher dimensional case, because of the IR behaviour of the fermion propagators. One possible way out of this could be to consider non-standard fields such as, for instance, dipole fields. This, however, should require a re-derivation of the main results, since the structure of the model will, in principle, be different from the standard minimal coupling case. Results on this will be reported elsewhere.

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